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ON SUBHARMONIC FUNCTIONS DOMINATED BY CERTAIN FUNCTIONS

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ABSTRACT

Given two kinds of functions $f(X)$ and $h(y)$ defined on the m-dimensional Euclidean space R^m $(m \geqq 1)$ and the set of positive real numbers respectively, we give an estimation of growth of subharmonic functions $u(P)$ defined on R^{m+n} $(n \geqq 1)$ such that

$$u(P) \leqq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, where $\|Y\|$ denotes the usual norm of Y. Using an obtained result, we give a sharpened form of an ordinary Phragmén–Lindelöf theorem with respect to the generalized cylinder $D \times R^n$, with a bounded domain D in R^m.

1. Introduction

Let $X=(x_1, x_2, \ldots, x_k)$ denote a point in the k-dimensional Euclidean space R^k $(k \geqq 1)$ and $\|X\|$ denote the norm of X

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}.$$

The k-dimensional Lebesgue measure of a set S in R^k is denoted by $|S|$. With a non-negative measurable function $f(X)$ defined on R^m $(m \geqq 1)$, we associate a non-increasing function $\eta = F_f(\xi)$ on the interval $(0, +\infty)$ such that for every $t \geqq 0$ the m-dimensional measure $|S_f(t)|$ of the set

$$S_f(t) = \{X \in R^m \,|\, f(X) \geqq t\}$$

is equal to the one-dimensional Lebesgue measure of the set

$$\{\xi \,|\, 0 < \xi < +\infty, F_f(\xi) \geqq t\}.$$

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Such a function $F_f(\xi)$ is obtained by considering the inverse function of $\xi = |S_f(\eta)|$ and is uniquely determined except on a countable set.

Domar [5, Theorem 3] proved the following fact:

Let U_1 be an open set in R^m and denote the bounded set

$$\{Y = (y_1, \dots, y_n) \in R^n, a_i < y_i < b_i \ (i = 1, 2, \dots, n)\}$$

by F, where a_i and b_i are constants satisfying $a_i < b_i$ $(i = 1, 2, \dots, n)$. Let $f(X)$ be a non-negative measurable function on R^m satisfying

$$(1) \qquad \int_0^1 \xi^{-(m-1)/m} \log^+ F_f(\xi) d\xi < +\infty$$

and $u(X, Y)$ be a subharmonic function on

$$E = U_1 \times F = \{(X, Y) \in R^{m+n} \mid X \in U_1, Y \in F\}$$

such that

$$u(X, Y) \leqq f(X) \qquad \text{on } E.$$

Then

$$u(X, Y) \leqq K$$

on any subset of E which is situated at a distance δ from the boundary of E, where K is a constant depending only on $f(X)$, δ and $b_i - a_i$ $(i = 1, 2, \dots, n)$.

A similar theorem was proved in Yoshida [11] in terms of spherical coordinates and applied to answer a question raised by Hayman [7; 3.6].

In the present paper, we first give a result extending Domar's result in the sense that

(i) in place of F, any open set U_2 $(U_2 \subset R^n)$ can be taken,

(ii) given a non-negative measurable function $f(X)$ on R^m satisfying (1) and a certain function $g(Y)$ defined on U_2, a subharmonic function $u(X, Y)$ dominated by $f(X)g(Y)$ on $U_0 = U_1 \times U_2$ is also dominated by $Kg(Y)$ with a constant K independent of $u(X, Y)$, on any subset of U_0 located at a positive distance from the boundary of U_0.

In connection with the growth property of $g(Y)$, we consider the special case where R^m, $\{Y \in R^n \mid \|Y\| > y_0\}$ ($y_0 \geqq 0$ is a constant) and a function free from rotation are taken as U_1, U_2 and $g(Y)$, respectively.

We next prove an ordinary Phragmén–Lindelöf theorem generalized in a satisfactory form, which extends a result of Deny and Lelong ([3, Théorème 2] and also [4, Théorème 2]) and a result of Brawn [1, Theorem 1].

Using the results thus obtained, we finally give a type of Phragmén–Lindelöf theorem which is different from the ordinary type and falls under the same category as a result in Wolf [10, Lemma] and a result in Yoshida [11, Corollary 3].

I wish to thank Professor M. Ohtsuka for his continuous advice and a referee for his valuable suggestions.

2. Statements of fundamental results

The proofs of all theorems in this section will be given in Section 4. The boundary of a set S in R^k and the distance between two sets S_1 and S_2 in R^k are denoted by ∂S and $\mathrm{dis}(S_1, S_2)$, respectively. We denote the k-dimensional closed ball having a center $P \in R^k$ and a radius r by $C_k(P, r)$. Let U be an open set in R^k ($k \geqq 1$). A function $\psi(P)$ defined on U and satisfying

$$\inf_{P \in U} \psi(P) > 0$$

is said to *grow regularly*, if there is a constant $\mu \geqq 1$ such that

$$\psi(P) \leqq \mu \psi(P_0)$$

for every $P_0 \in U$ and every $P \in C_k(P_0, 1) \cap U$.

The following result is obtained from Domar's result stated in Section 1.

THEOREM 1. *Let U_1 and U_2 be open sets in R^m ($m \geqq 1$) and R^n ($n \geqq 1$), respectively. Let $f(X)$ be a non-negative measurable function on R^m satisfying* (1) *and $g(Y)$ be a regularly growing function on U_2. Suppose that $u(P)$ is a subharmonic function on $U_0 = U_1 \times U_2$ such that*

$$u(P) \leqq f(X) g(Y)$$

for any $P = (X, Y) \in U_0$. Then, for any $\varepsilon > 0$, there exists a constant K dependent only on $f(X)$, μ and ε such that

$$u(P) \leqq K g(Y)$$

for every $P = (X, Y) \in U_0$, $\mathrm{dis}(P, \partial U_0) > \varepsilon$.

Let $y_0 \geqq 0$ be a constant and $h(y)$ be a regularly growing function defined on $(y_0, +\infty)$. It is easily seen that $h(\| Y \|)$ defined on $\{Y \in R^n \mid \| Y \| > y_0\}$ grows regularly. If we put $U_1 = R^m$, $U_2 = \{Y \in R^n \mid \| Y \| > y_0\}$, $g(Y) = h(\| Y \|)$ and $\varepsilon = 1$ in Theorem 1, we immediately have

THEOREM 2. *Let $f(X)$ be a non-negative measurable function on R^m satisfying* (1) *and $h(y)$ be a regularly gorwing function on $(y_0, +\infty)$, where $y_0 \geqq 0$ is a constant. Suppose that $u(X, Y)$ is a subharmonic function on $R^m \times R^n$ such that*

$$u(X, Y) \leqq f(X)h(\| Y \|)$$

for any $(X, Y) \in R^m \times R^n$, $\| Y \| > y_0$. Then, there exists a constant K dependent only on $f(X)$ and μ such that

$$u(X, Y) \leqq Kh(\| Y \|)$$

for every $(X, Y) \in R^m \times R^n$, $\| Y \| > y_0 + 1$.

REMARK. If a function $h(y)$ on $(y_0, +\infty)$ grows regularly, there are two positive constants A and B such that

$$h(y) \leqq Ae^{By} \qquad (y > y_0).$$

In fact, let y, $y > y_0$, be any number and take a non-negative integer n satisfying

$$n \leqq y - y_0 < n + 1.$$

Then

$$h(y) \leqq \mu h(y_0 + n) \leqq \mu^n h(y_0 + 1) \leqq \mu^{(y - y_0)} h(y_0 + 1) = Ae^{By},$$

where

$$A = \mu^{-y_0} h(y_0 + 1), \qquad B = \log \mu.$$

It follows from Remark that $h(y)$ in Theorem 2 must satisfy the growth condition

$$h(y) = O(e^{By}) \qquad (y \to +\infty) \tag{2}$$

for some constant $B > 0$. The following Theorem 3 is a generalization of Ohtsuka's [6], which shows that (2) is almost sharp.

THEOREM 3. *For any $\varepsilon > 0$, there exist a subharmonic function $u_\varepsilon(X, Y)$ on $R^m \times R^n$ satisfying*

$$\sup_{R^m \times R^n} u_\varepsilon(X, Y) \exp(-\| Y \|^{1+\varepsilon}) = +\infty \tag{3}$$

and a non-negative measurable function $f(X)$ on R^m satisfying (1) *such that*

$$u_\varepsilon(X, Y) \leqq f_\varepsilon(X) \exp(\| Y \|^{1+\varepsilon}) \qquad \text{on } R^m \times R^n. \tag{4}$$

QUESTION 1. The function $h(y)=\exp(y^{1+\varepsilon})$ does not satisfy (2). So we ask: Is Theorem 2 still true, even if the regular growth condition of $h(y)$ is replaced by the weaker condition (2)?

The following Theorem 4 shows that the exponent $-(m-1)/m$ in (1) is best possible in Theorems 1 and 2. Our example is essentially a multi-dimensional variant of Domar's [5, Theorem 5].

THEOREM 4. *There exist a subharmonic function $u(X,Y)$ on $R^m \times R^n$, a non-negative measurable function $f(X)$ on R^m satisfying*

$$\int_0^1 \xi^{-l} \log^+ F_f(\xi) d\xi < +\infty \tag{5}$$

for any $l<(m-1)/m$, and a regularly growing function $h(y)$ on $(0,+\infty)$ such that

$$u(X,Y) \leqq f(X)h(\|Y\|) \qquad on\ R^m \times (R^n - \{0\}) \tag{6}$$

and

$$\sup_{R^m \times (R^n - \{0\})} u(X,Y)h(\|Y\|)^{-1} = +\infty. \tag{7}$$

QUESTION 2. The non-negative measurable function $f(X)$ on R^m which we shall give to prove Theorem 4 has the property

$$K_1 \xi^{-1/m} \leqq \log^+ F_f(\xi) \leqq K_2 \xi^{-1/m}$$

for sufficiently small $\varepsilon > 0$, where K_1 and K_2 are two positive constants. So, H. Aikawa asks: *Is it possible to find a subharmonic function $u(X,Y)$ on $R^m \times R^n$, a non-negative measurable function $f(X)$ on R^m satisfying*

$$\log^+ F_f(\xi) = o(\xi^{-1/m}) \qquad (\xi \to 0) \tag{8}$$

and a regularly growing function $h(y)$ on $(0,+\infty)$ such that (6) *and* (7) *are satisfied*? If this question is negatively answered, we have an interesting result that (1) is replaced by (8) in Theorem 2.

3. Extended Phragmén–Lindelöf theorems

By R^+, we denote the set of all positive real numbers. Let G be a domain in R^k $(k \geqq 2)$. When a function $u(P)$ on G is given, we say that $u(P)$ *satisfies the Phragmén–Lindelöf boundary condition on ∂G, if*

$$\overline{\lim_{P \in G, P \to Q}} u(P) \leqq 0$$

for every $Q \in \partial G$. When two domains G_1 and G_2,

$$G_1 \subset R^m \quad (m \geqq 1), \qquad G_2 \subset R^n \quad (n \geqq 1),$$

and a function $u(X, Y)$ on $G_1 \times G_2$ are given, the maximum modulus $M(u, y)$ of $u(X, Y)$ is defined by

$$M(u, y) = \sup_{(X, Y) \in G_1 \times G_2, \|Y\| = y} u(X, Y) \qquad (y > 0).$$

Hardy and Rogosinski [6, Theorem 3] proved:

THEOREM HR. *Let I be an open interval (α, β) and $u(z)$ be a subharmonic function on the half-strip*

$$\Lambda = \{z = X + iY \mid X \in I, Y \in R^+\}$$

such that $u(z)$ satisfies the Phragmén–Lindelöf boundary condition on $\partial\Lambda$ and

$$\overline{\lim_{y \to \infty}} M(u, y) \exp\{-(\beta - \alpha)^{-1} \pi y\} \leqq 0.$$

Then

$$u(z) \leqq 0 \qquad (z \in \Lambda).$$

Deny and Lelong [3, Théorème 2], [4, Théorème 2] generalized Theorem HR to a function defined on a half-cylinder in the Euclidean space of higher dimension. In the following, a bounded domain in R^m having sufficiently smooth boundary (if $m = 1$, an open interval) is called a *bounded regular domain*. For a given bounded regular domain D, let $\lambda_D > 0$ be the first eigenvalue of the boundary value problem with respect to D:

$$\Delta f + \lambda_D f = 0 \quad \text{on } D, \qquad f = 0 \quad \text{on } \partial D,$$

where Δ denotes the Laplace operator (if $m = 1$, $\Delta = d^2/dx^2$). If D is an interval (α, β), we easily see

$$\sqrt{\lambda_D} = (\beta - \alpha)^{-1} \pi.$$

THEOREM DL. *Let D be a bounded regular domain in R^m $(m \geqq 1)$ and $u(P)$ be a subharmonic function on $\Gamma = D \times R^+$ such that $u(P)$ satisfies the Phragmén–Lindelöf boundary condition on $\partial\Gamma$ and*

$$\overline{\lim_{y \to \infty}} M(u, y) \exp(-\sqrt{\lambda_D} y) \leqq 0.$$

Then

$$u(P) \leqq 0 \qquad (P \in \Gamma).$$

On the other hand, Brawn [1, Theorem 1] generalized Theorem HR to a subharmonic function on the strip $(0,1) \times R^n$ in R^{n+1} $(n \geqq 1)$.

THEOREM B. *Let $u(P)$ be a subharmonic function on*

$$\Omega = (0,1) \times R^n \qquad (n \geqq 1)$$

such that $u(P)$ satisfies the Phragmén–Lindelöf boundary condition on $\partial\Omega$ and

$$\varlimsup_{y \to \infty} M(u, y) y^{(n-1)/2} \exp(-\pi y) \leqq 0.$$

Then

$$u(P) \leqq 0 \qquad (P \in \Omega).$$

The following Theorem 5 generalizes both Theorem DL and Theorem B.

THEOREM 5. *Let D be a bounded regular domain in R^m $(m \geqq 1)$ and $u(P)$ be a subharmonic function on $\Pi = D \times R^n$ $(n \geqq 1)$ such that $u(P)$ satisfies the Phragmén–Lindelöf boundary condition on $\partial\Pi$ and*

$$\varlimsup_{y \to \infty} M(u, y) y^{(n-1)/2} \exp(-\sqrt{\lambda_D} y) \leqq 0.$$

Then

$$u(P) \leqq 0 \qquad (P \in \Pi).$$

We now state the main result in this paper which further sharpens Theorem 5. It is the result based on Theorems 2 and 5.

THEOREM 6. *Let D be a bounded regular domain in R^m $(m \geqq 1)$ and $u(X, Y)$ be a subharmonic function on $\Pi = D \times R^n$ $(n \geqq 1)$ satisfying the Phragmén–Lindelöf boundary condition on $\partial\Pi$. Suppose that for a non-negative measurable function $f(X)$ on R^m satisfying* (1)

$$u(X, Y) \leqq \varepsilon(\| y \|) f(X) \| Y \|^{(1-n)/2} \exp(\sqrt{\lambda_D} \| Y \|) \qquad \text{on } D \times (R^n - \{0\})$$

where $\varepsilon(t)$ is a function on R^+ decreasing to 0 as $t \to +\infty$. Then

$$u(X, Y) \leqq 0 \qquad \text{on } \Pi.$$

The following Theorem 7 shows that the exponent $-(m-1)/m$ in (1) is best possible in Theorem 6.

THEOREM 7. *Let*

$$D_0 = \{X \in R^m \mid \|X\| < 2^{-1}\pi\} \qquad (m \geqq 1).$$

Then there exist an unbounded subharmonic function $u(X, Y)$ *on* $\Pi_0 = D_0 \times R^n$ $(n \geqq 1)$ *satisfying the Phragmén–Lindelöf boundary condition on* $\partial\Pi_0$, *a function* $\varepsilon(t)$ *on* R^+ *decreasing to* 0 *as* $t \to +\infty$ *and a non-negative measurable function* $f(X)$ *on* R^m *satisfying* (5) *for any* $l < (m-1)/m$, *such that*

$$u(X, Y) \leqq \varepsilon(\|Y\|) f(X) \|Y\|^{(1-n)/2} \exp(\sqrt{\lambda_{D_0}} \|Y\|) \qquad on\ D_0 \times (R^n - \{0\}). \tag{9}$$

The proofs of Theorems 5, 6 and 7 will be given in Section 5.

4. Proofs of Theorems 1, 3 and 4

PROOF OF THEOREM 1. (This is the proof suggested by a referee.) Let μ, $\mu \geqq 1$, be a constant such that

$$g(Y) \leqq \mu g(Y_0) \tag{10}$$

for any $Y_0 \in U_2$ and any $Y \in C_n(Y_0, 1) \cap U_2$. Take any ε, $0 < \varepsilon < 2$, and any point $P_0 = (X_0, Y_0) \in U_0$, $\mathrm{dis}(P_0, \partial U_0) > \varepsilon$. From (10) we have

$$u(P) \leqq \mu g(Y_0) f(X)$$

for any $P = (X, Y) \in C_{m+n}(P_0, 1) \cap U_0$. Consider the subharmonic function $u(P)\{\mu g(Y_0)\}^{-1}$ on the set

$$\Phi(P_0) = \{P = (X, Y) \in U_0 \mid \|X - X_0\| < \varepsilon/\sqrt{8}, |y_i - y_i^0| < \varepsilon/\sqrt{8n}\ (i = 1, 2, \ldots, n)\}$$

where $Y = (y_1, y_2, \ldots, y_n)$ and $Y_0 = (y_1^0, y_2^0, \ldots, y_n^0)$. If we apply Domar's result stated in Section 1, we obtain a constant K^* independent of P_0 and $u(X, Y)$ such that

$$u(P) \leqq K^* \mu g(Y_0)$$

for every $P \in \Phi(P_0)$ and hence

$$u(P_0) \leqq K^* \mu g(Y_0).$$

Putting $K = K^* \mu$, we finally have that

$$u(P) \leqq K g(Y)$$

for every $P = (X, Y) \in U_0$, $\mathrm{dis}(P, \partial U_0) > \varepsilon$.

PROOF OF THEOREM 3. Given any $\varepsilon > 0$, consider $u_\varepsilon^*(X, Y)$ on $R^m \times R^n$ defined by

$$u_\varepsilon^*(X,Y) = \begin{cases} \|Y\|^\varepsilon (\cos\|X\|)\exp(\|Y\|^{1+\varepsilon} - \|X\|^2\|Y\|^\varepsilon) \\ \qquad \text{on } \{(X,Y) \mid X \in R^m, \|X\| < 2^{-1}\pi, Y \in R^n\} \\ 0 \qquad \text{elsewhere.} \end{cases}$$

If we write $\|X\| = x$, $\|Y\| = y$ and

$$g(x,y) = \exp(y^{1+\varepsilon} - x^2 y^\varepsilon)$$

for simplicity, we have

$$\Delta u_\varepsilon^* = \frac{\partial^2 u_\varepsilon^*}{\partial x^2} + \frac{m-1}{x}\frac{\partial u_\varepsilon^*}{\partial x} + \frac{n-1}{y}\frac{\partial u_\varepsilon^*}{\partial y} + \frac{\partial^2 u_\varepsilon^*}{\partial y^2}$$

$$\geqq g(x,y)[y^{3\varepsilon}\{(1+\varepsilon)^2 - o(1)\}\cos x + y^{2\varepsilon}\{4 - x^{-2}(m-1)y^{-\varepsilon}\}]x \sin x$$

$$\geqq \begin{cases} g(x,y)[y^{3\varepsilon}\{(1+\varepsilon)^2 - o(1)\}\cos x + y^{2\varepsilon}\{2^{-1}\sqrt{2} - o(1)\}] \\ \qquad (4^{-1}\pi \leqq x \leqq 2^{-1}\pi,\ y \to +\infty) \\ g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2 - o(1) - (m-1)y^{-2\varepsilon}x^{-1}\sin x\} \\ \qquad \geqq g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2 - o(1)\} \quad (0 < x \leqq 4^{-1}\pi,\ y \to +\infty) \end{cases}$$

by an elementary computation. This shows that $u^*(X, Y)$ is subharmonic on $R^m \times \{Y \in R^n \mid \|Y\| > a\}$ for a sufficiently large a.

Choose a constant M_ε so that

$$u_\varepsilon^*(X,Y) \leqq M_\varepsilon$$

on $R^m \times \{Y \in R^n \mid \|Y\| < 2a\}$. Define

$$u_\varepsilon(X,Y) = \max\{u_\varepsilon^*(X,Y), M_\varepsilon\} \qquad \text{on } R^m \times R^n$$

and note that u_ε is subharmonic on $R^m \times R^n$. Put

(11) $$f_\varepsilon(X) = \max\{\|X\|^{-2}, M_\varepsilon\} \qquad \text{on } R^m.$$

Since

$$F_{f_\varepsilon}(\xi) = (A_m \xi^{-1})^{2/m} \qquad (\xi < A_m M_\varepsilon^{-m/2}),$$

$f_\varepsilon(X)$ satisfies (1). We observe (3), because

$$u_\varepsilon(0,Y)\exp(-y^{1+\varepsilon}) \to +\infty \qquad \text{as } y \to +\infty.$$

We next see that

$$\exp(x^2 y^\varepsilon) > x^2 y^\varepsilon \qquad \text{on } R^+ \times R^+$$

and hence

$$x^{-2} > y^{\varepsilon} \exp(-x^2 y^{\varepsilon}) \qquad \text{on } R^+ \times R^+.$$

From this fact and (11), (4) follows immediately.

PROOF OF THEOREM 4. Put

$$v(X, Y) = \exp(e^{\|Y\|}\cos\|X\|)\cos(e^{\|Y\|}\sin\|X\|) \qquad \text{on } R^m \times R^n$$

and consider

$$u^*(X, Y) = \{v(X, Y)\}^{2m}.$$

If we write $\|X\| = x$ and $\|Y\| = y$, we have

$$\Delta u^* = 2mv^{2m-2}\left[(2m-1)\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + v\Delta v\right]$$

$$= 2mv^{2m-2}\exp(y + 2e^y \cos x)q(x, y)$$

where

$q(x, y) =$

$$(2m-1)e^y + \cos(e^y \sin x)\left\{\frac{n-1}{y}\cos(x + e^y \sin x) - \frac{m-1}{x}\sin(x + e^y \sin x)\right\}.$$

Since

$$\sin(x + e^y \sin x) \leqq x + e^y \sin x \leqq x(1 + e^y) \qquad (0 < x < 2^{-1}\pi),$$

we see that

$$q(x, y) \geqq me^y - (m-1) - (n-1)y^{-1}$$

if $0 < x < 2^{-1}\pi$, $e^y \sin x < 2^{-1}\pi$. Hence, for a sufficiently large y_0, $y_0 > \log(2^{-1}\pi)$, we have

$$\Delta u^* \geqq 0$$

on the set

$$S = \{(X, Y) \in R^m \times R^n \mid \|X\| < \pi/2, e^{\|Y\|}\sin\|X\| < \pi/2, \|Y\| > y_0\}.$$

Let

$$D_0 = \{X \in R^m \mid \|X\| < \pi/2\}.$$

Choose a positive constant M such that

$$u^*(X, Y) \leqq M$$

on $D_0 \times \{Y \in R^n \mid \|Y\| < 2y_0\}$ and define a subharmonic function $u(X, Y)$ on $R^m \times R^n$ by

$$u(X,Y) = \begin{cases} M^{-1}\max\{u^*(X,Y), M\} & \text{on } S, \\ 1 & \text{elsewhere.} \end{cases}$$

We define $f(X)$ on R^m by

$$f(X) = \sup_{Y \in R^n} u(X, Y) \tag{12}$$

and $h(y)$ on R^+ by

$$h(y) \equiv 1.$$

It is evident that $h(y)$ is a regularly growing function on R^+ and (6) holds. Since

$$u(0, Y) = M^{-1}\exp(2me^{\|Y\|})$$

at any $Y \in R^n$ having sufficiently large $\|Y\|$, we have (7).

Finally, we shall show that (5) holds for any l, $l < (m-1)/m$. Put

$$v^*(x, y) = \exp(e^y \cos x)\cos(e^y \sin x)$$

for $x \in R$ and $y \in R$, $y > y_0$. Then, for any fixed y, $v^*(x, y)$ increases from 0 to $\exp(e^y)$ as x decreases from $\sin^{-1}(2^{-1}\pi e^{-y})$ to 0. This fact gives that

$$u(X, Y) > M^{-1}t^{2m}$$

on the domain surrounded by the set

$$\{(X, Y) \in S \mid v(X, Y) = t\}$$

for a sufficiently large t. For a given t, consider the curve

$$L = \{(x, y) \in R^2 \mid v^*(x, y) = t, 0 \leqq x < \pi/2\}$$

in the plane and put

$$x^* = \max_{(x,y) \in L} x.$$

Since

$$\frac{dy}{dx} = \tan(x + e^y \sin x)$$

along L, we have

$$x^* + e^{y^*}\sin x^* = \pi/2, \qquad (x^*, y^*) \in L.$$

Hence, x^* satisfies

$$\sin x^* \exp\{(2^{-1}\pi - x^*)\cot x^*\} = t.$$

Since

$$|S_f(M^{-1}t^{2m})| = A_m x^{*m}$$

from (12), we have

$$F_f(\xi) = M^{-1}[\sin\{(A_m^{-1}\xi)^{1/m}\}]^{2m} \exp[2m\{2^{-1}\pi - (A_m^{-1}\xi)^{1/m}\}\cot\{(A_m^{-1}\xi)^{1/m}\}].$$

Thus, for a sufficiently small $\varepsilon > 0$,

$$K_1\xi^{-1/m} \leqq \log F_f(\xi) \leqq K_2\xi^{-1/m}$$

where K_1 and K_2 are two positive constants. This gives (5) for any $l < (m-1)/m$.

5. Proofs of Theorems 5, 6 and 7

PROOF OF THEOREM 5. This proof is based on both methods used to prove Theorem DL and Theorem B. For a given bounded regular domain D, we denote the positive eigenfunction corresponding to the eigenvalue λ_D by $f_D(X)$.

Define $h_D(X, Y)$ on Π by

$$h_D(X, Y) = f_D(X)\|Y\|^{1-n/2} I_{n/2-1}(\sqrt{\lambda_D}\|Y\|),$$

where $I_{n/2-1}(y)$ is the Bessel function of the third kind, of order $n/2-1$ (see e.g. Watson [8, p. 77]). It is easy to see that $h_D(X, Y)$ is harmonic on Π. We also remark that

$$I_{n/2-1}(y) = (2\pi y)^{-1/2} e^y (1 + o(1)) \qquad (y \to +\infty)$$

(see Watson [8, p. 203]).

First, consider a subharmonic function $u_1(P)$ on Π defined by

$$u_1(P) = u(P) - \eta_1 h_D(P) \qquad (\eta_1 > 0). \tag{13}$$

Take a closed ball $B \subset D$ and choose a positive constant ε_1 such that

$$f_D(X) \geqq \varepsilon_1 \qquad \text{on } B.$$

If we choose a positive constant y_1 such that

$$M(u, y) < 2^{-1}\varepsilon_1\eta_1 C_D y^{(1-n)/2}\exp(\sqrt{\lambda_D}y) \qquad (y \geqq y_1)$$

where

$$C_D = (2\pi\sqrt{\lambda_D})^{-1/2},$$

we see that

$$u_1(P) \leqq \varepsilon_1\eta_1 C_D\{-2^{-1} - o(1)\}\|Y\|^{(1-n)/2}\exp(\sqrt{\lambda_D}\|Y\|)$$

for any $P = (X, Y)$, $X \in B$, $\|Y\| \geqq y_1$. Hence, there are a value M and a point $P_0 \in B \times R^n$ such that

$$u_1(P_0) = M \quad \text{and} \quad u_1(P) \leqq M \qquad \text{on } B \times R^n. \tag{14}$$

Next, by using the properties of monotonicity and continuity of the eigenvalues (e.g. see Courant and Hilbert [2, Theorem 3 on p. 409 and Theorem 10 on p. 421]), take a bounded regular domain D^*, $D^* \subset R^m$ such that

$$(D - B) \cup \partial(D - B) \subset D^* \quad \text{and} \quad \lambda_D < \lambda_{D^*} < \lambda_{D-B}.$$

Consider a subharmonic function $u_2(P)$ on $(D - B) \times R^n$ defined by

$$u_2(P) = u_1(P) - \eta_2 h_{D^*}(P) \qquad (\eta_2 > 0). \tag{15}$$

If we take a positive number ε_2 such that

$$f_{D^*}(X) \geqq \varepsilon_2 \qquad \text{on } (D - B) \cup \partial(D - B)$$

and a number y_2 such that

$$M(u, y) < \varepsilon_2\eta_2 C_{D^*} y^{(1-n)/2}\exp(\sqrt{\lambda_D}y) \qquad (y \geqq y_2),$$

we have that

$$\begin{aligned} u_2(P) &\leqq u(P) - \eta_2 h_{D^*}(P) \\ &\leqq \varepsilon_2\eta_2 C_{D^*}\|Y\|^{(1-n)/2}[\exp\{(\sqrt{\lambda_D} - \sqrt{\lambda_{D^*}})\|Y\|\} - (1 + o(1))] \end{aligned}$$

for any $P = (X, Y) \in (D - B) \times R^n$, $\|Y\| \geqq y_2$. Hence, with (14) the maximal principle gives that

$$u_2(P) \leqq \max(0, M) \qquad \text{on } (D - B) \times R^n.$$

We also have that

$$u_1(P) \leqq \max(0, M) \qquad \text{on } (D - B) \times R^n,$$

because η_2 is chosen arbitrarily small in (15). Further, we have from (14) that

$$u_1(P) \leqq \max(0, M) \qquad \text{on } D \times R^n.$$

The maximal principle and (14) give that $M \leqq 0$ and hence

$$u_1(P) \leqq 0 \qquad \text{on } D \times R^n.$$

Letting $\eta_1 \to 0$ in (13), we conclude that

$$u(P) \leqq 0 \qquad \text{on } D \times R^n.$$

PROOF OF THEOREM 6. For each positive integer m, take a number t_m such that

$$\varepsilon(t) \leqq 1/m \qquad (t \geqq t_m).$$

Then

$$u(X, Y) \leqq f(X)\{m^{-1} \| Y \|^{(1-n)/2} \exp(\sqrt{\lambda_D} \| Y \|)\}$$

at every $(X, Y) \in D \times \{Y \in R^n \mid \| Y \| \geqq t_m\}$. If we put

$$h_m(y) = m^{-1} y^{(1-n)/2} \exp(\sqrt{\lambda_D} y),$$

we easily see that

$$h_m(y+1) \leqq h_m(y) \exp(\sqrt{\lambda_D} y) \qquad (y > t_m).$$

Hence, if we also put $u(X, Y) = 0$ on $R^m \times R^n - \Pi$ and apply Theorem 2, we can find a constant K independent of m such that

$$u(X, Y) \leqq K h_m(\| Y \|) = K m^{-1} \| Y \|^{(1-n)/2} \exp(\sqrt{\lambda_D} \| Y \|)$$

for every $(X, Y) \in D \times \{Y \in R^n \mid \| Y \| > t_m + 1\}$. This gives that

$$\overline{\lim_{y \to \infty}} M(u, y) y^{(n-1)/2} \exp(-\sqrt{\lambda_D} y) \leqq 0.$$

The conclusion follows from Theorem 5.

PROOF OF THEOREM 7. For the subharmonic function $u(X, Y)$ taken in the proof of Theorem 4, consider a function

$$u(X, Y) - 1 \qquad \text{on } \Pi_0 = D_0 \times R^n.$$

Representing this function by $u(X, Y)$ again, we easily see that it satisfies the Phragmén-Lindelöf boundary condition on $\partial \Pi_0$. Define $f(X)$ on R^m by

$$f(X) = \begin{cases} \sup_{Y \in R^n} u(X, Y) & \text{on } D_0, \\ 0 & \text{elsewhere.} \end{cases}$$

Then we can show (5) for any $l < (m-1)/m$ as in the proof of Theorem 4. If we define $\varepsilon(t)$ on R^+ by

$$\varepsilon(t) = t^{(n-1)/2}\exp(-\sqrt{\lambda_{D_0}}\,t),$$

we evidently obtain that (9) holds for these $f(X)$ and $\varepsilon(t)$.

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